Residual Systems and Proof Terms

for Formalizing Confluence Criteria

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Motivation

Residual Systems

- study orthogonality
- of steps instead of systems
- abstract from term rewriting setting

Proof Terms

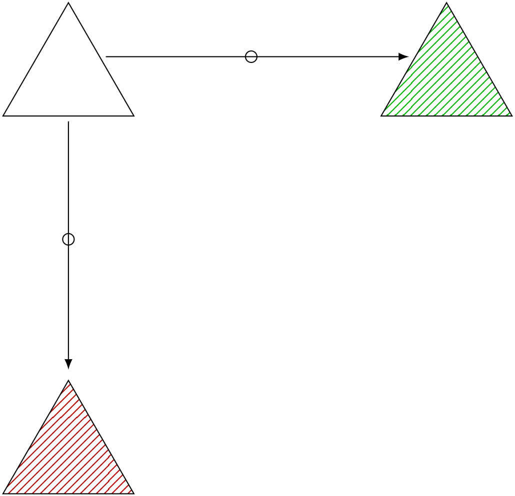
- how to represent rewrite steps?
- avoid syntactic accidents
- use proof terms for Meseguer's rewriting logic

Both

- facilitate formalization in proof assistant

Theorem (van Oostrom, '97)

If $\mathcal{R}$ is left-linear and $t \mathrel{\circ\!\!\!\rightarrow} u$ for all critical peaks $t \leftarrow s \rightarrow_\epsilon u$ then $\mathrel{\circ\!\!\!\rightarrow}$ has the diamond property.



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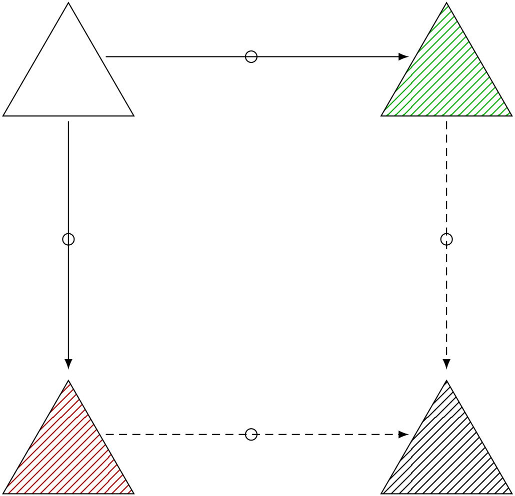

Theorem (van Oostrom, '97)

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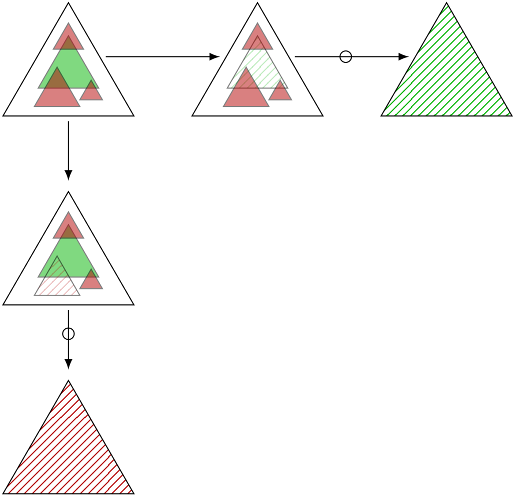

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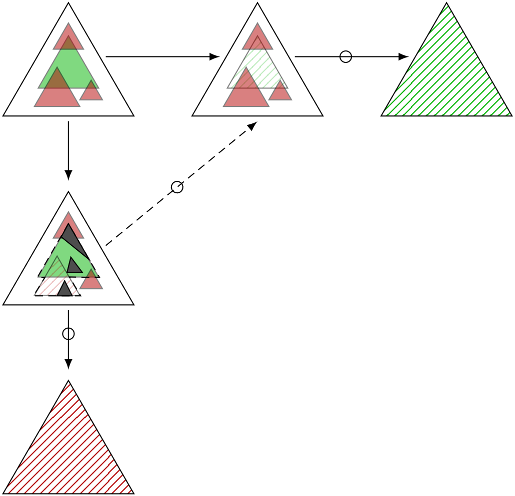

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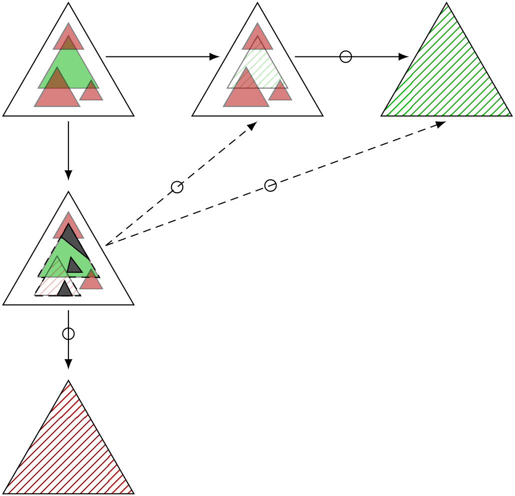
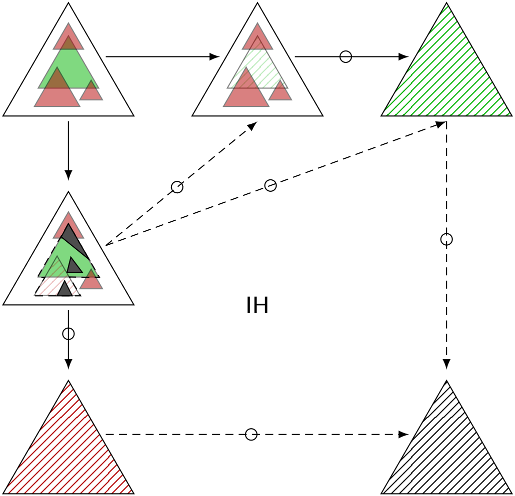


Theorem (van Oostrom, '97)

If $\mathcal{R}$ is left-linear and $t \nrightarrow u$ for all critical peaks $t \leftarrow s \rightarrow_\epsilon u$ then $\nrightarrow$ has the diamond property.



Focusing on Steps

Definition

abstract reduction system is structure $(\mathcal{A}, \Phi, \mathsf{src}, \mathsf{tgt})$ with

- $\mathcal{A}$ is set of objects and Φ is set of steps
- $\mathsf{src} : \Phi \to \mathcal{A}$ and $\mathsf{tgt} : \Phi \to \mathcal{A}$ are source and target functions

Focusing on Steps

$(\mathcal{A}, \{(\mathsf{src}(\phi), \mathsf{tgt}(\phi)) \mid \phi \in \Phi\})$ is abstract rewrite system

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Definition

- let $\mathcal{R}$ be TRS over signature $\mathcal{F}$
- $\mathsf{var}(\ell)$ denotes sequence of variables in ℓ in some fixed order
- $(s_1, \ldots, s_n)_\ell$ denotes substitution $\{x_i \mapsto s_i \mid 1 \leqslant i \leqslant n\}$ for $\mathsf{var}(\ell) = (x_1, \ldots, x_n)$

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- for each rule $\ell \to r \in \mathcal{R}$ introduce fresh rule symbol $\underline{\ell \to r}$ with $\mathsf{ar}(\underline{\ell \to r}) = |\mathsf{var}(\ell)|$
- proof terms $\mathcal{PT}(\mathcal{F}, \mathcal{R})$ are terms over $\mathcal{F}$ and rule symbols

Definition

src and tgt for proof terms are defined by

$$\mathsf{src}(x) = x \qquad \mathsf{tgt}(x) = x$$

$$\mathsf{src}(f(A_1, \dots, A_n) = f(\mathsf{src}(A_1), \dots, \mathsf{src}(A_n))$$

$$\mathsf{tgt}(f(A_1, \dots, A_n) = f(\mathsf{tgt}(A_1), \dots, \mathsf{tgt}(A_n))$$

$$\mathsf{src}(\underline{\ell \to r}(A_1, \dots, A_n)) = \ell(\mathsf{src}(A_1), \dots, \mathsf{src}(A_n))_\ell$$

$$\mathsf{tgt}(\underline{\ell \to r}(A_1, \dots, A_n)) = r(\mathsf{tgt}(A_1), \dots, \mathsf{tgt}(A_n))_\ell$$

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$$x \mathrel{\circ\!\!\!\to}_x x$$

$$f(s_1,\ldots,s_n) \mathrel{\circ\!\!\!\to}_{f(A_1,\ldots,A_n)} f(t_1,\ldots,t_n)$$

$$\ell(s_1,\ldots,s_n)_\ell \mathrel{\circ\!\!\!\to}_{\underline{\ell\to r}(A_1,\ldots,A_n)} r(t_1,\ldots,t_n)_\ell$$

$s_i \mathrel{\circ\!\!\!\to}_{A_i} t_i$ for all $1 \leqslant i \leqslant n$

Abstract Residual Systems

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abstract residual system is tuple $(\mathcal{A}, \Phi, \mathsf{src}, \mathsf{tgt}, 1, \backslash)$ with

- $(\mathcal{A}, \Phi, \mathsf{src}, \mathsf{tgt})$ is abstract reduction system
- $1 : \mathcal{A} \to \Phi$ is unit function with $\mathsf{src}(1(a)) = \mathsf{tgt}(1(a)) = a$
- $\backslash : \{(\phi, \psi) \mid \mathsf{src}(\phi) = \mathsf{src}(\psi)\} \to \Phi$ is residuation function with $\mathsf{src}(\psi \setminus \phi) = \mathsf{tgt}(\phi)$ and $\mathsf{tgt}(\psi \setminus \phi) = \mathsf{tgt}(\phi \setminus \psi)$

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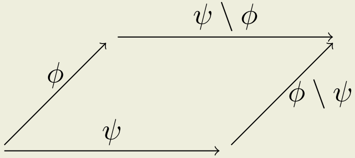

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$$\phi \setminus \phi = 1 \qquad \phi \setminus 1 = \phi \qquad 1 \setminus \phi = 1$$

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Question: are the residual identities independent?

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Example

natural numbers with cut-off subtraction

$$(n \dot{-} m) \dot{-} (k \dot{-} m) = n \dot{-} max(m, k) = (n \dot{-} k) \dot{-} (m \dot{-} k)$$

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Example

(multi)sets with (multi)set difference

$$(A - B) - (C - B) = A - (B \cup C) = (A - C) - (B - C)$$

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Example

multistep rewriting with proof terms as steps

Definition

for co-initial proof terms A, B the residual operation $A \setminus B$ is defined by

$$
\begin{aligned}
x \setminus x &= x \\
f(A_1, \ldots, A_n) \setminus f(B_1, \ldots, B_n) &= f(A_1 \setminus B_1, \ldots, A_n \setminus B_n) \\
\underline{\ell{\to}r}(A_1, \ldots, A_n) \setminus \underline{\ell{\to}r}(B_1, \ldots, B_n) &= r(A_1 \setminus B_1, \ldots, A_n \setminus B_n)_\ell \\
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Example

$$1 : \mathsf{b} \to \mathsf{a} \qquad 2 : \mathsf{g}(x) \to \mathsf{h}(x, x)$$

$$\mathsf{f}(\mathsf{g}(\mathsf{b})) \twoheadrightarrow\!\!\!\!\circ_{\mathsf{f}(\underline{2}(\underline{1}))} \mathsf{f}(\mathsf{h}(\mathsf{a}, \mathsf{a}))$$

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- projection order is defined by $\phi \lesssim \psi$ if $\phi \setminus \psi = 1$
- projection equivalence is defined by $\phi \simeq \psi$ if $\phi \lesssim \psi$ and $\psi \lesssim \phi$

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$\lesssim$ *is quasi-order and* $\simeq$ *is equivalence relation*

Proof

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Joining Steps

Definition

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$$\phi \lesssim \phi \sqcup \psi,\ \psi \lesssim \phi \sqcup \psi,\ \phi \lesssim \chi \wedge \psi \lesssim \chi \Longrightarrow \phi \sqcup \psi \lesssim \chi$$

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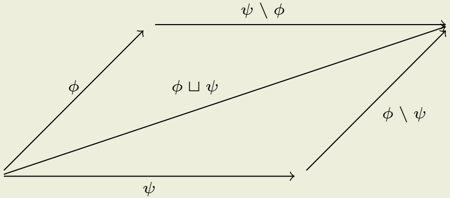
Definition

for co-initial proof terms A, B join $A \sqcup B$ is defined by

$$
\begin{aligned}
x \sqcup x &= x \\
f(A_1, \dots, A_n) \sqcup f(B_1, \dots, B_n) &= f(A_1 \sqcup B_1, \dots, A_n \sqcup B_n) \\
\underline{\ell \to r}(A_1, \dots, A_n) \sqcup \underline{\ell \to r}(B_1, \dots, B_n) &= \underline{\ell \to r}(A_1 \sqcup B_1, \dots, A_n \sqcup B_n) \\
\underline{\ell \to r}(A_1, \dots, A_n) \sqcup \ell(B_1, \dots, B_n)_\ell &= \underline{\ell \to r}(A_1 \sqcup B_1, \dots, A_n \sqcup B_n) \\
\ell(A_1, \dots, A_n)_\ell \sqcup \underline{\ell \to r}(B_1, \dots, B_n) &= \underline{\ell \to r}(A_1 \sqcup B_1, \dots, A_n \sqcup B_n)
\end{aligned}
$$

Residual Cube



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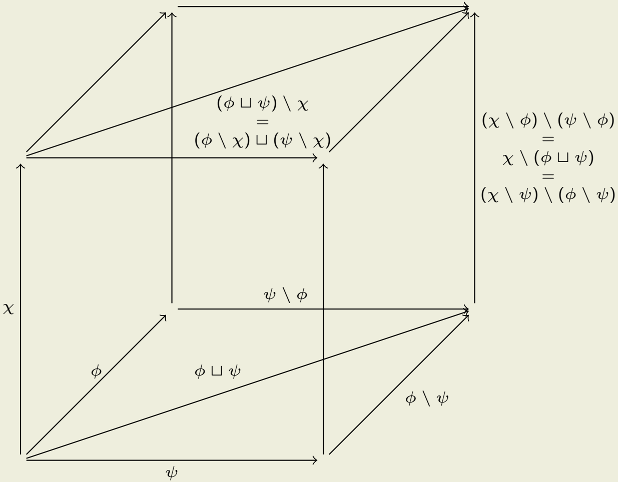

Residual Systems with Composition

Definition

- steps ϕ and ψ are composable if $\mathsf{tgt}(\phi) = \mathsf{src}(\psi)$
- residual system with composition is residual system with additional binary function ; on composable steps such that

$$(\phi \,;\, \psi) \setminus \chi = ((\phi \setminus \chi) \,;\, (\psi \setminus (\chi \setminus \phi)))$$
$$1 \,;\, 1 = 1 \qquad \chi \setminus (\phi \,;\, \psi) = (\chi \setminus \phi) \setminus \psi$$

- designated join is $\phi \sqcup_d \psi = \phi \,;\, (\psi \setminus \phi)$

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- designated join is $\phi \sqcup_d \psi = \phi \,;\, (\psi \setminus \phi)$

Lemma

if the underlying residual system has joins then they are projection equivalent to the designated joins in the residual system with composition

Proof

$$(\phi \sqcup \psi) \setminus (\phi \sqcup_d \psi) = (\phi \sqcup \psi) \setminus (\phi \,;\, (\psi \setminus \phi))$$

Proof

$$\begin{aligned}(\phi \sqcup \psi) \setminus (\phi \sqcup_d \psi) &= (\phi \sqcup \psi) \setminus (\phi \,;\, (\psi \setminus \phi)) \\ &= ((\phi \sqcup \psi) \setminus \phi) \setminus (\psi \setminus \phi)\end{aligned}$$

Proof

$$\begin{aligned}(\phi \sqcup \psi) \setminus (\phi \sqcup_d \psi) &= (\phi \sqcup \psi) \setminus (\phi \,;\, (\psi \setminus \phi)) \\ &= ((\phi \sqcup \psi) \setminus \phi) \setminus (\psi \setminus \phi) \\ &= ((\phi \setminus \phi) \sqcup (\psi \setminus \phi)) \setminus (\psi \setminus \phi)\end{aligned}$$

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$$\begin{aligned}(\phi \sqcup \psi) \setminus (\phi \sqcup_d \psi) &= (\phi \sqcup \psi) \setminus (\phi \,;\, (\psi \setminus \phi)) \\ &= ((\phi \sqcup \psi) \setminus \phi) \setminus (\psi \setminus \phi) \\ &= ((\phi \setminus \phi) \sqcup (\psi \setminus \phi)) \setminus (\psi \setminus \phi) \\ &= (1 \sqcup \psi \setminus \phi) \setminus (\psi \setminus \phi)\end{aligned}$$

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\begin{aligned}
(\phi \sqcup \psi) \setminus (\phi \sqcup_d \psi) &= (\phi \sqcup \psi) \setminus (\phi \,;\, (\psi \setminus \phi)) \\
&= ((\phi \sqcup \psi) \setminus \phi) \setminus (\psi \setminus \phi) \\
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$$(\phi \sqcup_d \psi) \setminus (\phi \sqcup \psi) = (\phi \,;\, (\psi \setminus \phi)) \setminus (\phi \sqcup \psi)$$

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(\phi \sqcup \psi) \setminus (\phi \sqcup_d \psi) &= (\phi \sqcup \psi) \setminus (\phi \mathbin{;} (\psi \setminus \phi)) \\
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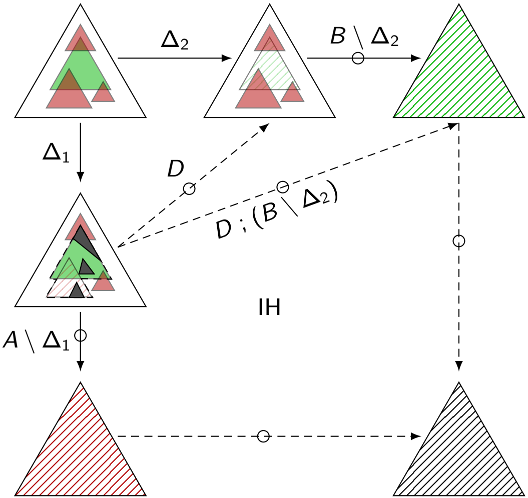
\end{aligned}
$$

Claim

by left-linearity and having picked innermost overlap:

$$(B - \Delta_2) \setminus \Delta_2 = (B - \Delta_2) \setminus (\Delta_1 \; ; \; D)$$



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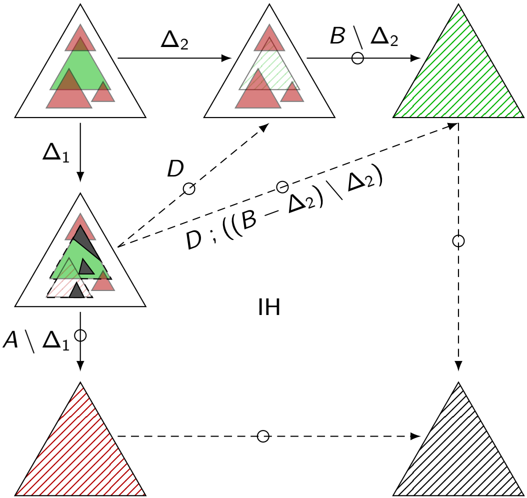

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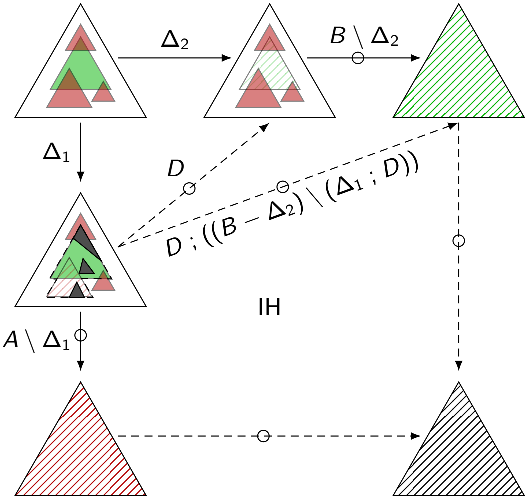

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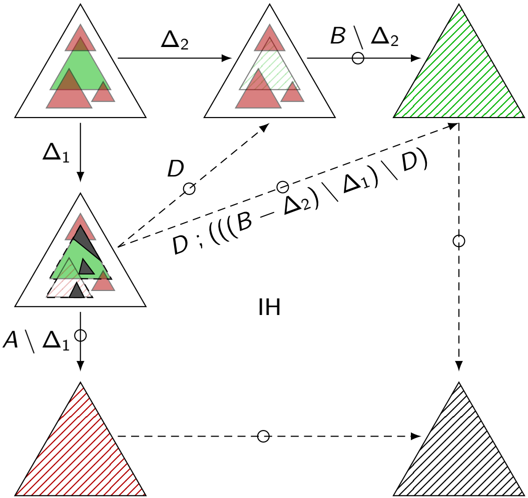

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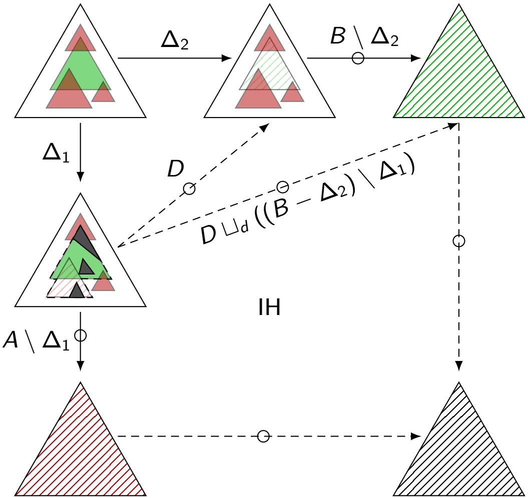

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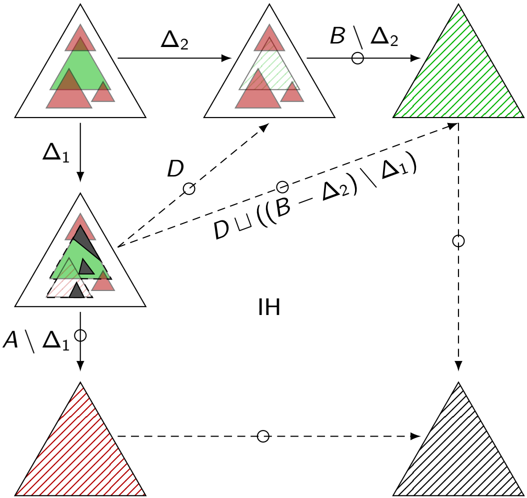

Conclusion

- use proof terms to reason about steps
- use residual theory for abstract algebraic reasoning
- manage challenges of formalization in proof assistant