Improving Automatic Confluence Analysis of Rewrite Systems by Redundant Rules

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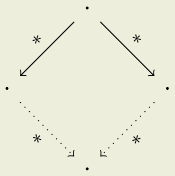
Outline

- Motivation
- Redundant Rules
- Illustrating Examples
- Experimental Results
- Conclusion

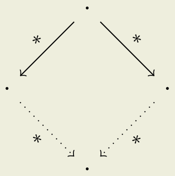
Automatic Confluence Analysis

Confluence



Automatic Confluence Analysis

Confluence Criteria



Knuth and Bendix

Automatic Confluence Analysis

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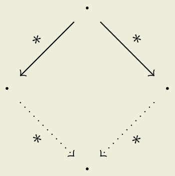

Knuth and Bendix, orthogonality

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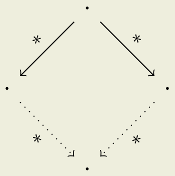

Knuth and Bendix, orthogonality, strongly/parallel/development closed critical pairs

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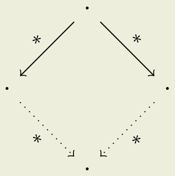

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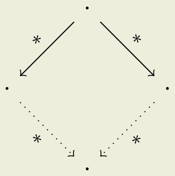

Knuth and Bendix, orthogonality, strongly/parallel/development closed critical pairs, decreasing diagrams (rule labeling), parallel and simultaneous critical pairs, divide and conquer techniques (commutation, layer preservation, order-sorted decomposition), decision procedures, depth/weight preservation, reduction-preserving completion, Church-Rosser modulo, relative termination and extended critical pairs, non-confluence techniques (tcap, tree automata, interpretation), ...

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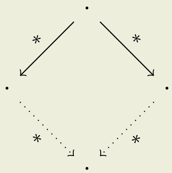

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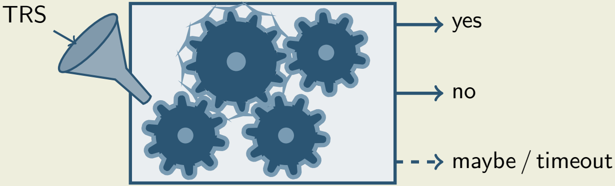
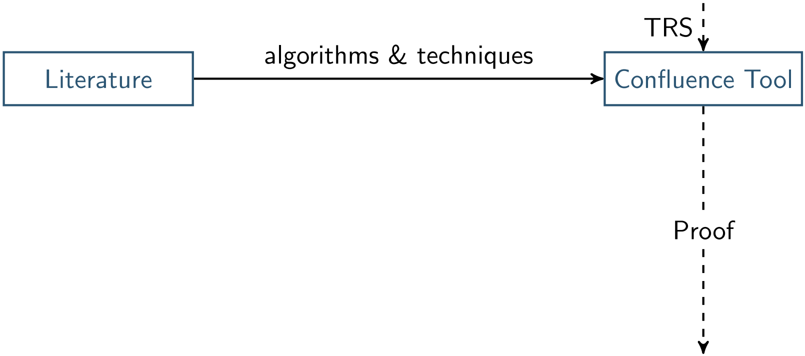
Automation



Formalization & Certification



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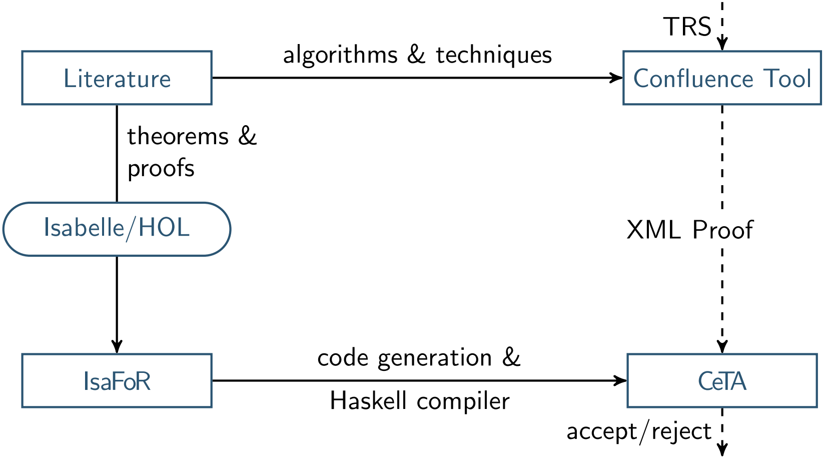

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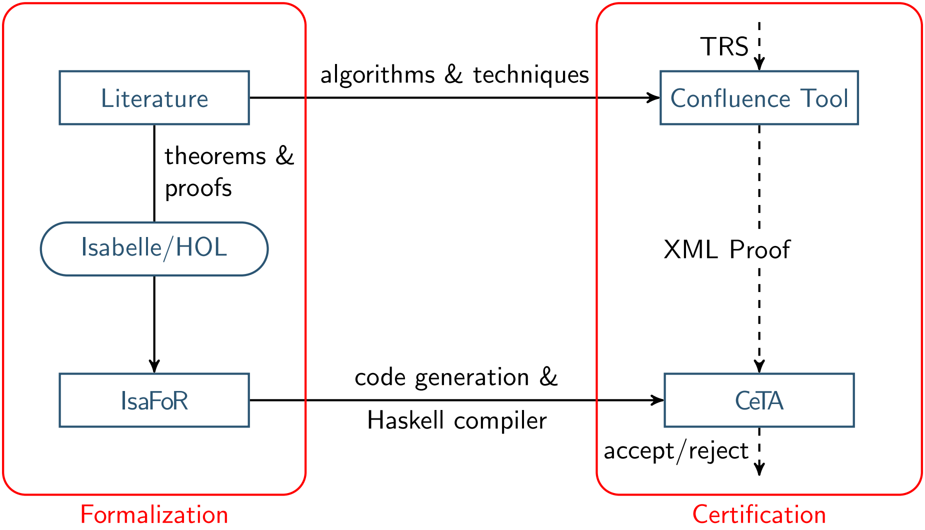

Motivating Examples

Example (Shintani, 41st TRS meeting)

- TRS $\mathcal{R}$ $\quad \mathsf{f}(\mathsf{f}(x)) \to x \qquad \mathsf{f}(x) \to \mathsf{f}(\mathsf{f}(x))$

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- thus also $\mathcal{R}$ is confluent

Example (Felgenhauer, IWC 2012)

- TRS $\mathcal{R}$

$$
\begin{array}{llll}
\mathsf{f}(\mathsf{g}(\mathsf{a}),\mathsf{g}(y)) \to \mathsf{b} & \mathsf{f}(x,y) \to \mathsf{f}(x,\mathsf{g}(y)) & \mathsf{g}(x) \to x & \mathsf{a} \to \mathsf{g}(\mathsf{a}) \\
\mathsf{f}(\mathsf{h}(x),\mathsf{h}(\mathsf{a})) \to \mathsf{c} & \mathsf{f}(x,y) \to \mathsf{f}(\mathsf{h}(x),y) & \mathsf{h}(x) \to x & \mathsf{a} \to \mathsf{h}(\mathsf{a})
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- additional rules can be simulated by $\mathcal{R}$ and thus also $\mathcal{R}$ is non-confluent

Theory

Lemma

if $\ell \to_{\mathcal{R}}^{*} r$ *for every rule* $\ell \to r \in \mathcal{S}$ *then* $\to_{\mathcal{R}}^{*} = \to_{\mathcal{R} \cup \mathcal{S}}^{*}$

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- $\ell \to_{\mathcal{R}}^* r$ from assumption

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- $\ell \to_{\mathcal{R}}^* r$ from assumption
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Corollary

if $\ell \to_{\mathcal{R}}^* r$ *for every rule* $\ell \to r \in \mathcal{S}$ *then* $\mathcal{R}$ *is confluent if and only if* $\mathcal{R} \cup \mathcal{S}$ *is confluent*

Lemma

if $\ell \leftrightarrow^*_{\mathcal{R}} r$ *for every rule* $\ell \to r \in \mathcal{S}$ *then* $\leftrightarrow^*_{\mathcal{R} \cup \mathcal{S}} = \leftrightarrow^*_{\mathcal{R}}$

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Corollary

if $\mathcal{R}$ *is confluent and* $\ell \leftrightarrow^*_{\mathcal{R}} r$ *for every rule* $\ell \to r \in \mathcal{S}$ *then* $\mathcal{R} \cup \mathcal{S}$ *is confluent*

Removing Rules

Example (Gramlich/Lucas, RTA 2006; Hirokawa/Middeldorp, JAR 2011)

- TRS $\mathcal{R}$

$$\begin{array}{rrr} \mathsf{hd}(x : y) \to x & \mathsf{nats} \to 0 : \mathsf{inc}(\mathsf{nats}) & \mathsf{inc}(x : y) \to \mathsf{s}(x) : \mathsf{inc}(y) \\ \mathsf{tl}(x : y) \to y & \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) & \end{array}$$

Removing Rules

Example (Gramlich/Lucas, RTA 2006; Hirokawa/Middeldorp, JAR 2011)

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- $\mathcal{R}$ without $\mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))$ is orthogonal and thus confluent

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- $\mathcal{R}$ without $\mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))$ is orthogonal and thus confluent
- since

$$\begin{aligned} \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) &\to \mathsf{inc}(\mathsf{tl}(0 : \mathsf{inc}(\mathsf{nats}))) \to \mathsf{inc}(\mathsf{inc}(\mathsf{nats})) \\ &\leftarrow \mathsf{tl}(\mathsf{s}(0) : \mathsf{inc}(\mathsf{inc}(\mathsf{nats}))) \leftarrow \mathsf{tl}(\mathsf{inc}(0 : \mathsf{inc}(\mathsf{nat}))) \\ &\leftarrow \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \end{aligned}$$

also $\mathcal{R}$ is confluent

Removing Rules

Example (Suzuki/Aoto/Toyama, Computer Software 2013)

- TRS $\mathcal{R}$

$$\begin{aligned} \mathsf{f}(x,x) &\to \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x)) & \qquad \mathsf{f}(x,y) &\to \mathsf{f}(\mathsf{h}(x),\mathsf{h}(y)) \\ \mathsf{g}(x) &\to \mathsf{p}(x) & \mathsf{h}(x) &\to \mathsf{p}(x) \end{aligned}$$

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$$\mathsf{f}(x,x) \to \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x)) \qquad \mathsf{f}(x,y) \to \mathsf{f}(\mathsf{h}(x),\mathsf{h}(y))$$
$$\mathsf{g}(x) \to \mathsf{p}(x) \qquad \mathsf{h}(x) \to \mathsf{p}(x)$$

- $\mathcal{R}$ without $\mathsf{f}(x,x) \to \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x))$ is orthogonal and thus confluent
- since $\mathsf{f}(x,x) \downarrow \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x))$ using remaining rules

$$\begin{aligned} \mathsf{f}(x,x) &\to \mathsf{f}(\mathsf{h}(x),\mathsf{h}(x)) \to \mathsf{f}(\mathsf{p}(x),\mathsf{h}(x)) \\ &\to \mathsf{f}(\mathsf{p}(x),\mathsf{p}(x)) \leftarrow \mathsf{f}(\mathsf{g}(x),\mathsf{p}(x)) \leftarrow \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x)) \end{aligned}$$

$\mathcal{R}$ is also confluent

Removing and Adding Rules

Example (Aoto/Toyama/Uchida 2014, Cop 412)

- TRS $\mathcal{R}$

$$\mathsf{f}(x, y) \to \mathsf{f}(\mathsf{g}(x), \mathsf{g}(x)) \qquad \mathsf{f}(x, x) \to \mathsf{a} \qquad \mathsf{g}(x) \to x$$

Removing and Adding Rules

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- first add $\mathsf{f}(x,y) \to \mathsf{a}$

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- next remove $\mathsf{f}(x,y) \to \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x))$ and $\mathsf{f}(x,x) \to \mathsf{a}$

Removing and Adding Rules

Example (Aoto/Toyama/Uchida 2014, Cop 412)

- TRS $\mathcal{R}$

$$\mathsf{f}(x,y) \to \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x)) \qquad \mathsf{f}(x,x) \to \mathsf{a} \qquad \mathsf{g}(x) \to x$$

- first add $\mathsf{f}(x,y) \to \mathsf{a}$
- next remove $\mathsf{f}(x,y) \to \mathsf{f}(\mathsf{g}(x),\mathsf{g}(x))$ and $\mathsf{f}(x,x) \to \mathsf{a}$
- resulting TRS is orthogonal and hence $\mathcal{R}$ is confluent

Implementation and Experiments

Strategies

- add (minimal) joining sequences of critical pairs as rules

$$\mathcal{S} \subseteq \{s \to u, t \to u \mid s \leftarrow\!\rtimes\!\rightarrow t \text{ with } s \to_{\mathcal{R}}^{*} u \text{ and } t \to_{\mathcal{R}}^{*} u\}$$

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- shorten joining sequences by rewriting right-hand sides of rules

$$\mathcal{S} = \{\ell \to t \mid \ell \to r \in \mathcal{R} \text{ and } r \to_{\mathcal{R}} t\}$$

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$$\mathcal{S} = \{\ell \to r \mid \ell \downarrow_{\mathcal{R}} r\}$$

- add geared towards specific confluence criterion (e.g. development closed)

$$\mathcal{S} = \{s \to t \mid s \leftarrow\!\rtimes\!\rightarrow t \text{ with } s \to_{\mathcal{R}}^* t \text{ and } s \not\mathrel{+\!\!\!\!\!\to}_{\mathcal{R}} t\}$$

Implementation and Experiments

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- add (minimal) joining sequences of critical pairs as rules

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$$\mathcal{S} = \{\ell \to r \mid \ell \downarrow_{\mathcal{R}} r\}$$

- add reversed reversible rules

$$\mathcal{S} = \{r \to \ell \mid \ell \to r \in \mathcal{R} \text{ with } r \to_{\mathcal{R}}^* \ell\}$$

Implementation and Experiments

Strategies

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$$\mathcal{S} \subseteq \{s \to u, t \to u \mid s \leftarrow\!\rtimes\!\to t \text{ with } s \to_{\mathcal{R}}^{*} u \text{ and } t \to_{\mathcal{R}}^{*} u\}$$

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- delete rules whose sides are joinable by other rules

$$\mathcal{S} = \{\ell \to r \mid \ell \downarrow_{\mathcal{R}} r\}$$

- delete rules whose sides are convertible by other rules

$$\mathcal{S} = \{\ell \to r \mid \ell \downarrow_{\mathcal{R} \cup \mathcal{R}^{-1}} r\}$$

276 TRSs in Confluence Problem Database

	CSI
yes	155
no	47
maybe/timeout	74

276 TRSs in Cops

	CSI	CSI_{js}
yes	155	156
no	47	48
maybe/timeout	74	72

Strategies

js add minimal joining sequences of critical pairs as rules
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276 TRSs in Cops

	CSI	CSI_{js}	CSI_{rhs}
yes	155	156	159
no	47	48	47
maybe/timeout	74	72	70

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276 TRSs in Cops

	CSI	CSI_{js}	CSI_{rhs}	CSI_{del}
yes	155	156	159	163
no	47	48	47	47
maybe/timeout	74	72	70	66

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276 TRSs in Cops

	CSI	CSI_{js}	CSI_{rhs}	CSI_{del}	CSI_{all}
yes	155	156	159	163	166
no	47	48	47	47	48
maybe/timeout	74	72	70	66	62

Strategies

- js: add minimal joining sequences of critical pairs as rules
- rhs: shorten joining sequences by rewriting right-hand sides of rules
- del: delete rules whose sides are joinable by other rules

Formalization and Certification

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- formalized as part of IsaFoR

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  - bound on length of derivations that show $\ell \to_{\mathcal{R}}^* r$ for all added rules, i.e., all $\ell \to r$ in $\mathcal{S} \setminus \mathcal{R}$
  - either bound on length of derivations that show $\ell \downarrow_{\mathcal{S}} r$ or explicit conversions $\ell \leftrightarrow_{\mathcal{S}}^* r$ for all deleted rules, i.e., all $\ell \to r$ in $\mathcal{R} \setminus \mathcal{S}$

276 TRSs in Cops

	CSI	$\mathrm{CSI}_{\mathrm{js}}$	$\mathrm{CSI}_{\mathrm{rhs}}$	$\mathrm{CSI}_{\mathrm{del}}$	$\mathrm{CSI}_{\mathrm{all}}$
yes	155	156	159	163	166
no	47	48	47	47	48
maybe/timeout	74	72	70	66	62

certified	✓CSI
yes	71
no	47
maybe/timeout	158

Strategies

- **js** add minimal joining sequences of critical pairs as rules
- **rhs** shorten joining sequences by rewriting right-hand sides of rules
- **del** delete rules whose sides are joinable by other rules

276 TRSs in Cops

	CSI	CSI_{js}	CSI_{rhs}	CSI_{del}	CSI_{all}
yes	155	156	159	163	166
no	47	48	47	47	48
maybe/timeout	74	72	70	66	62

certified	✓CSI	✓CSI_{js}	✓CSI_{rhs}	✓CSI_{del}	✓CSI_{all}
yes	71	86	73	78	104
no	47	48	47	47	48
maybe/timeout	158	142	156	151	124

Strategies

js add minimal joining sequences of critical pairs as rules

rhs shorten joining sequences by rewriting right-hand sides of rules

del delete rules whose sides are joinable by other rules

Related Work

- van Oostrom, 2014: feeble orthogonality
- Gramlich 2000; Zantema, 2005: rewrite right-hand sides (for termination)

Related Work

Definition

- TRS $\mathcal{R}$ is reversible if $\rightarrow_{\mathcal{R}} \;\subseteq\; {}_{\mathcal{R}}\!\overset{*}{\leftarrow}$
- $\mathcal{R}^{\pm} = \mathcal{R} \cup \mathcal{R}^{-1}$

Related Work

Definition

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Theorem (Aoto/Toyama, LMCS 2012)

for left-linear terminating $\mathcal{S}$ and reversible $\mathcal{P}$, if

- $\mathsf{CP}(\mathcal{S},\mathcal{S}) \;\subseteq\; \to_{\mathcal{S}}^{*} \cdot {}_{\mathcal{P}^{\pm}}\!\!\leftarrow\!\!\!\!\!\!\parallel\;\; \cdot {}_{\mathcal{S}}^{\;*}\!\!\leftarrow$
- $\mathsf{CP}_{\mathsf{in}}(\mathcal{P}^{\pm},\mathcal{S}) \;=\; \varnothing$
- $\mathsf{CP}(\mathcal{S},\mathcal{P}^{\pm}) \;\subseteq\; \to_{\mathcal{S}}^{*} \cdot \parallel\!\!\!\!\!\to_{\mathcal{P}^{\pm}} \cdot {}_{\mathcal{S}}^{\;*}\!\!\leftarrow$

then $\mathcal{S} \cup \mathcal{P}$ is confluent

Related Work

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for left-linear terminating $\mathcal{S}$ and reversible $\mathcal{P}$, if

- $\mathsf{CP}(\mathcal{S},\mathcal{S}) \subseteq \to_{\mathcal{S}}^{*} \cdot {}_{\mathcal{P}^{\pm}}\!\leftarrow\!\!\!\!\!\!\;+\!\!\!\!+ \cdot {}_{\mathcal{S}}^{*}\!\leftarrow$
- $\mathsf{CP}_{\mathsf{in}}(\mathcal{P}^{\pm},\mathcal{S}) = \varnothing$
- $\mathsf{CP}(\mathcal{S},\mathcal{P}^{\pm}) \subseteq \to_{\mathcal{S}}^{*} \cdot \,+\!\!\!\!+\!\!\!\to_{\mathcal{P}^{\pm}} \cdot {}_{\mathcal{S}}^{*}\!\leftarrow$

then $\mathcal{S} \cup \mathcal{P}$ is confluent

Reduction-Preserving Completion Procedure

$$\frac{\langle \mathcal{S} \cup \{\ell \to r\}, \mathcal{P}\rangle}{\langle \mathcal{S} \cup \{\ell \to r'\}, \mathcal{P}\rangle} \quad r \leftrightarrow_{\mathcal{P}}^{*} r' \qquad\qquad \frac{\langle \mathcal{S}, \mathcal{P}\rangle}{\langle \mathcal{S} \cup \{\ell \to r\}, \mathcal{P}\rangle} \quad \ell \leftrightarrow_{\mathcal{P}}^{*} \cdot \to_{\mathcal{S}}^{*} r$$

$$\frac{\langle \mathcal{S}, \mathcal{P}\rangle}{\langle \mathcal{S}', \mathcal{P}'\rangle} \quad \mathcal{S} \cup \mathcal{P} = \mathcal{S}' \cup \mathcal{P}' \text{ and } \mathcal{P}' \text{ is reversible}$$

Summary

Addition and Removal of Redundant Rules

- results in simpler and faster confluence proofs
- adds power to confluence tools
- is easy to formalize and certify
- boosts certifiable proofs