Certified Rule Labeling

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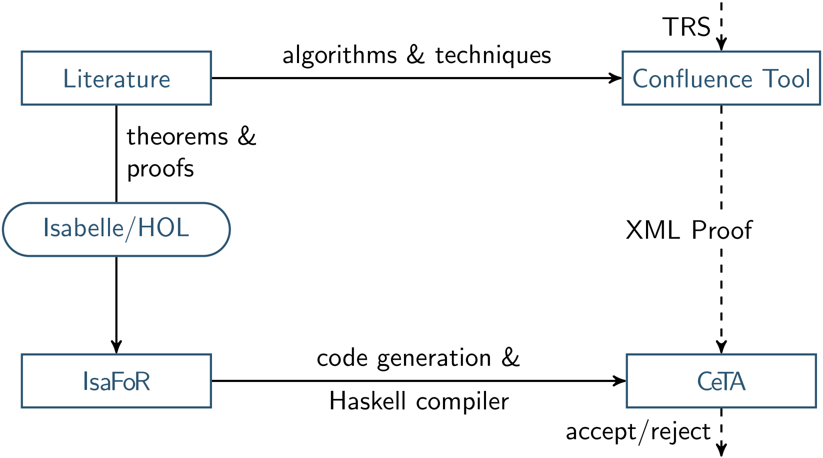
Overview

- Introduction
- Rule Labeling
- Relative Termination
- Certification
- Conclusion

Formalization & Certification



Decreasing Diagrams

Theorem (van Oostrom 1994)

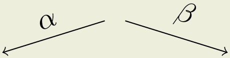
A locally decreasing ARS is confluent.

Decreasing Diagrams

Definition

An ARS $\{\rightarrow_\alpha\}_{\alpha \in \mathcal{I}}$ is locally decreasing if

- $\exists$ well-founded relation $<$ on $\mathcal{I}$ with



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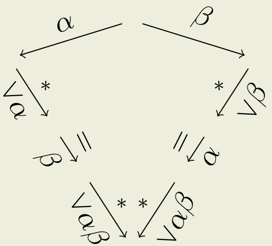

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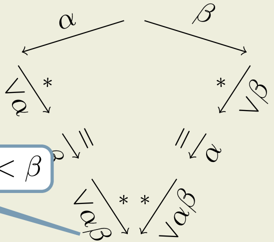

labels γ with $\gamma < \alpha$ or $\gamma < \beta$

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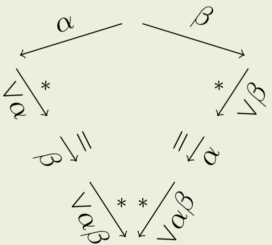

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A locally decreasing ARS is confluent.

- formalized by Zankl, RTA 2013

Contribution: Formalization

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A linear TRS is confluent if it is locally decreasing for the rule labeling.

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1. specialize decreasingness from ARSs to TRSs

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2. formalize rule labeling

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Theorem (van Oostrom, 2008)

A linear TRS is confluent if it is locally decreasing for the rule labeling.

Theorem (Zankl, Felgenhauer, Middeldorp, 2011)

A left-linear TRS $\mathcal{R}$ is confluent if $\mathcal{R}_d/\mathcal{R}_{nd}$ is terminating and all its critical peaks are decreasing for the rule labeling.

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1. specialize decreasingness from ARSs to TRSs
2. formalize rule labeling
3. formalize source labeling and interplay with rule labeling
4. check confluence proof certificates generated by automated tools

From ARSs to TRSs

Lemma

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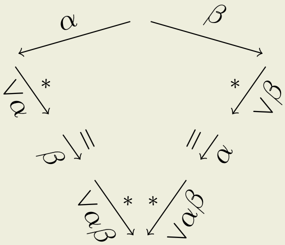

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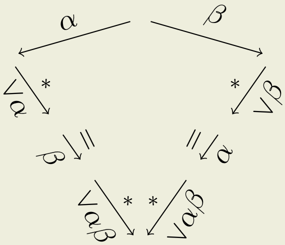

infinitely many!

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Local Peaks

$$s[r_1\sigma_1]_p \leftarrow s[l_1\sigma_1]_p = s = s[l_2\sigma_2]_q \rightarrow s[r_2\sigma_2]_q$$

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Local Peaks

$$s[r_1\sigma_1]_p \leftarrow s[l_1\sigma_1]_p = s = s[l_2\sigma_2]_q \rightarrow s[r_2\sigma_2]_q$$

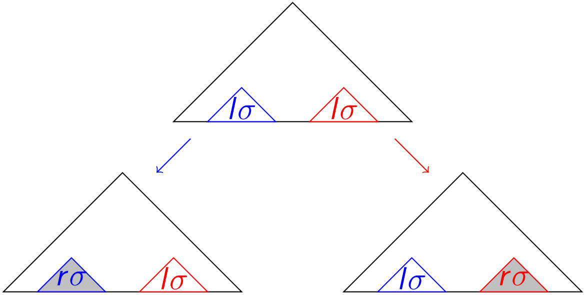
three possibilities (modulo symmetry):

(parallel peak) $p \parallel q$

(function peak) $q \leqslant p$ and $p \backslash q \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(l_2)$

(variable peak) $q \leqslant p$ and $p \backslash q \notin \mathcal{P}\mathrm{os}_{\mathcal{F}}(l_2)$

Local Peaks: Parallel Peak



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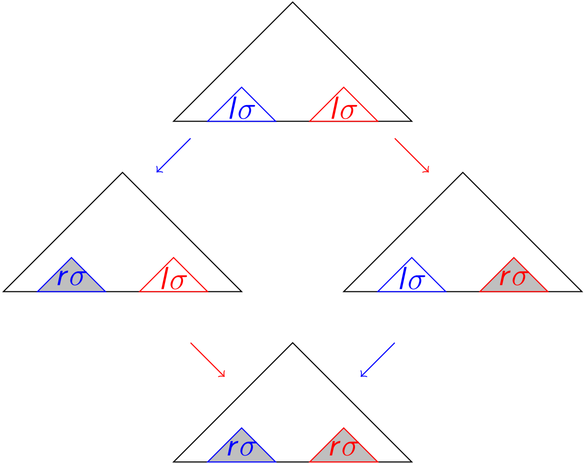
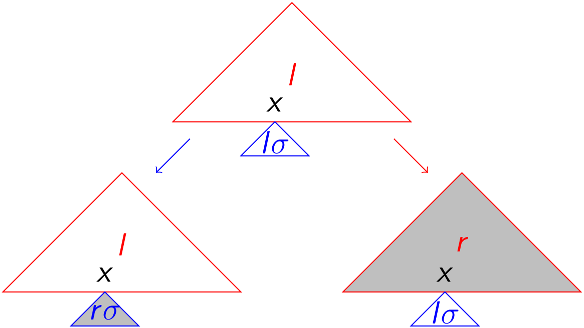


infinitely many

Local Peaks: Variable Peak ($l \to r$ linear)



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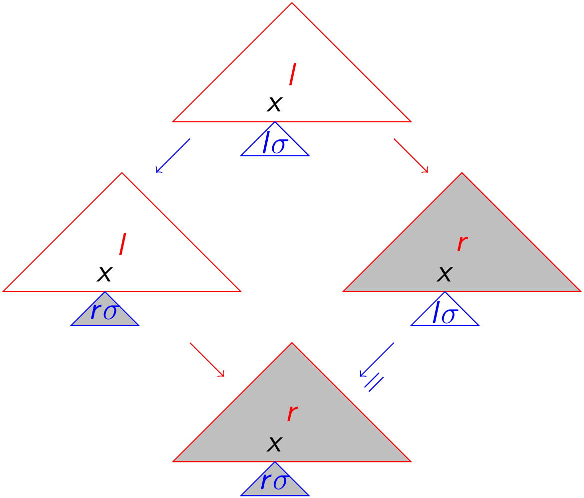
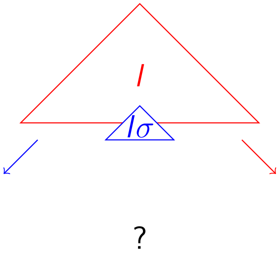


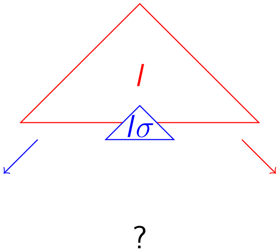
infinitely many

Local Peaks: Function Peak



infinitely many!

Local Peaks: (Instance of) Critical Peak



finitely representable!

Labeling

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A linear TRS which is locally decreasing for the rule labeling is confluent.

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- maps rewrite steps to labels
- is closed under contexts and substitutions

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- maps rewrite steps to labels
  $\ell(s \to t) = \alpha$
- is closed under contexts and substitutions
  If $\ell(s \to t) = \ell(u \to v)$ then $\ell(C[s\sigma] \to C[t\sigma]) = \ell(C[u\sigma] \to C[v\sigma])$
  If $\ell(s \to t) > \ell(u \to v)$ then $\ell(C[s\sigma] \to C[t\sigma]) > \ell(C[u\sigma] \to C[v\sigma])$

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- is compatible with $\mathcal{R}$ if parallel and variables peaks of $\mathcal{R}$ are locally decreasing for ℓ

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Proof

- ARS $\bigcup_\alpha \{(s,t) \mid s \to t \text{ and } \ell(s \to t) = \alpha\}$ is locally decreasing
- conclude by main result of decreasing diagrams

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$$\ell^i(s \to_{l \to r,p,\sigma} t) = i(l \to r) \qquad i \colon \mathcal{R} \to \mathbb{N}$$

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If $\mathcal{R}$ is linear then rule labeling is compatible labeling.

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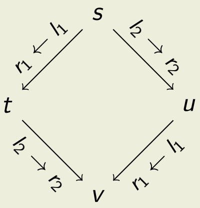
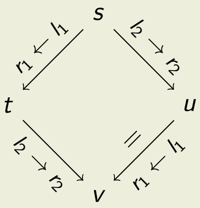


Parallel Peak



Variable Peak

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Theorem

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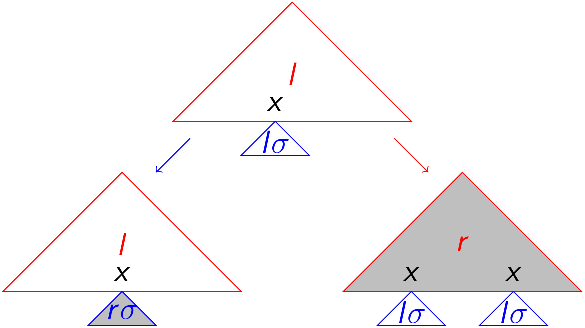
Local Peaks: Variable Peak ($l \to r$ left-linear)



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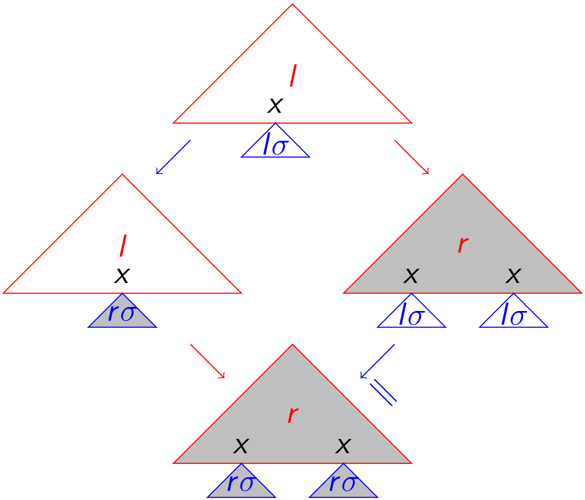

Source Labeling

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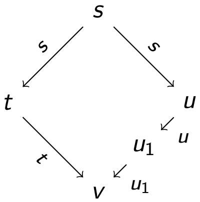
- $\ell^{\mathsf{src}}(s \to_{l \to r, p, \sigma} t) = s$
- labels are compared with $\to^{+}_{\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}}$



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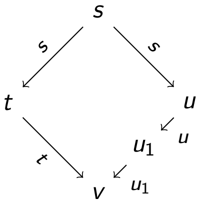

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An ARS $\{\to_\alpha\}_{\alpha \in \mathcal{I}}$ is extended locally decreasing if

- $\exists$ well-founded relation $<$ on $\mathcal{I}$ and preorder $\leqslant$ with $\leqslant \cdot < \cdot \leqslant \subseteq <$ and $_\alpha\!\leftarrow \cdot \to_\beta \subseteq \to^{*}_{\vee\alpha} \cdot \to^{=}_{\vee\!\!\vee\beta} \cdot \to^{*}_{\vee\alpha\beta} \cdot {}_{\vee\alpha\beta}^{\;*}\!\leftarrow \cdot {}_{\vee\!\!\vee\alpha}^{\;=}\!\leftarrow \cdot {}_{\vee\beta}^{\;*}\!\leftarrow$

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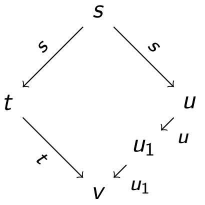

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Lemma (Hirokawa, Middeldorp 2010)

Every extended locally decreasing ARS is locally decreasing.

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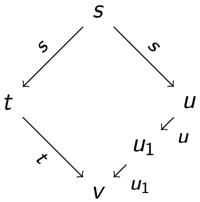
- labels are compared with $\to^{+}_{\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}}$ and $\to^{*}_{\mathcal{R}}$



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$\ell_1 \times \ell_2(s \to t) = (\ell_1(s \to t), \ell_2(s \to t))$
$(\alpha_1, \alpha_2) \geqslant (\beta_1, \beta_2)$ iff $\alpha_1 > \beta_1$ or $\alpha_1 \geqslant \beta_1$ and $\alpha_2 \geqslant \beta_2$
$(\alpha_1, \alpha_2) > (\beta_1, \beta_2)$ iff $\alpha_1 > \beta_1$ or $\alpha_1 \geqslant \beta_1$ and $\alpha_2 > \beta_2$

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Proof Sketch

- choose $\ell^{\mathsf{src}} \times \ell^i$ as labeling
- for critical peaks: along a rewrite sequence labels never increase with respect to ℓ^{src}

Certificates

Required Contents

- function $i \colon \mathcal{R} \to \mathbb{N}$
- certificate for termination of $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$
- joining sequences for critical peaks

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Observations

- CeTA has to compute critical peaks
- CeTA computes variants of critical peaks
- checking relative termination condition is completely independent of checking decreasingness for rule labeling

Experimental results

method	success
(weak) orthogonality	
Knuth-Bendix	
strong closedness	
ℓ^i	
$\ell^i + \mathsf{SN}(\mathcal{R}_\mathsf{d}/\mathcal{R}_\mathsf{nd})$	
$\sum$	

Table: Experimental results for 148 TRSs from CoCo 2014.

Experimental results

method	success	CoCo 2013
(weak) orthogonality	4	✓
Knuth-Bendix	26	✓
strong closedness	28	✓
ℓ^i		✗
$\ell^i + \mathsf{SN}(\mathcal{R}_\mathsf{d}/\mathcal{R}_\mathsf{nd})$		✗
$\sum$		45

Table: Experimental results for 148 TRSs from CoCo 2014.

Experimental results

method	success	CoCo 2013	CoCo 2014
(weak) orthogonality	4	✓	✓
Knuth-Bendix	26	✓	✓
strong closedness	28	✓	✓
ℓ^i	41	✗	✓
$\ell^i + \mathsf{SN}(\mathcal{R}_\mathsf{d}/\mathcal{R}_\mathsf{nd})$		✗	✗
$\sum$		45	56

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Experimental results

method	success	CoCo 2013	CoCo 2014	CeTA 2.19
(weak) orthogonality	4	✓	✓	✓
Knuth-Bendix	26	✓	✓	✓
strong closedness	28	✓	✓	✓
ℓ^i	41	✗	✓	✓
$\ell^i + \mathsf{SN}(\mathcal{R}_\mathsf{d}/\mathcal{R}_\mathsf{nd})$	46	✗	✗	✓
$\sum$		45	56	58

Table: Experimental results for 148 TRSs from CoCo 2014.

Conclusion

Summary

- formalization of rule labeling
- in combination with relative termination using source labeling
- checking confluence certificates based on decreasing diagrams for the first time

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- formalization of rule labeling
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Future Work

- lexicographic combination of labelings in certificate
- support more labelings
- label parallel/development steps