Formalization of Classical Confluence Results for Left-Linear Term Rewrite Systems

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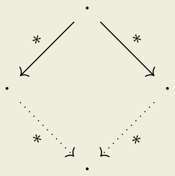
44th TRS Meeting
February 22–23, 2016

Outline

- Motivation
- Strongly Closed Critical Pairs
- Parallel Closed Critical Pairs
- Conclusion

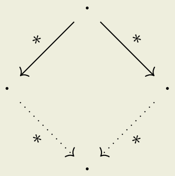
Automatic Confluence Analysis

Confluence



Automatic Confluence Analysis

Confluence Criteria



Knuth and Bendix

Automatic Confluence Analysis

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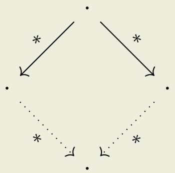
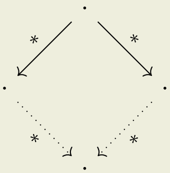


Knuth and Bendix, orthogonality

Automatic Confluence Analysis

Confluence Criteria



Knuth and Bendix, orthogonality, strongly/parallel/development closed critical pairs

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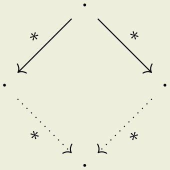

Knuth and Bendix, orthogonality, strongly/parallel/development closed critical pairs, decreasing diagrams (rule labeling)

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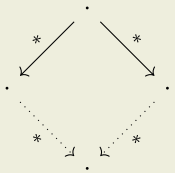

Knuth and Bendix, orthogonality, strongly/parallel/development closed critical pairs, decreasing diagrams (rule labeling), parallel and simultaneous critical pairs, divide and conquer techniques (commutation, layer preservation, order-sorted decomposition), decision procedures, depth/weight preservation, reduction-preserving completion, Church-Rosser modulo, relative termination and extended critical pairs, non-confluence techniques (tcap, tree automata, interpretation), . . .

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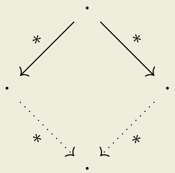

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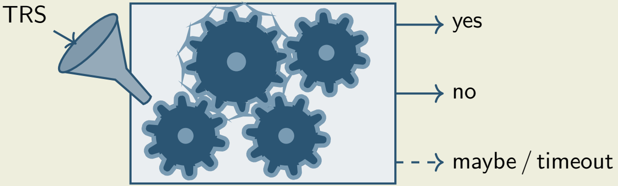
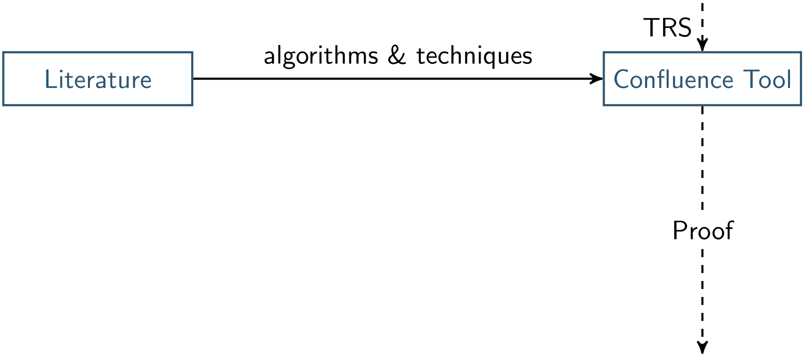


Formalization & Certification



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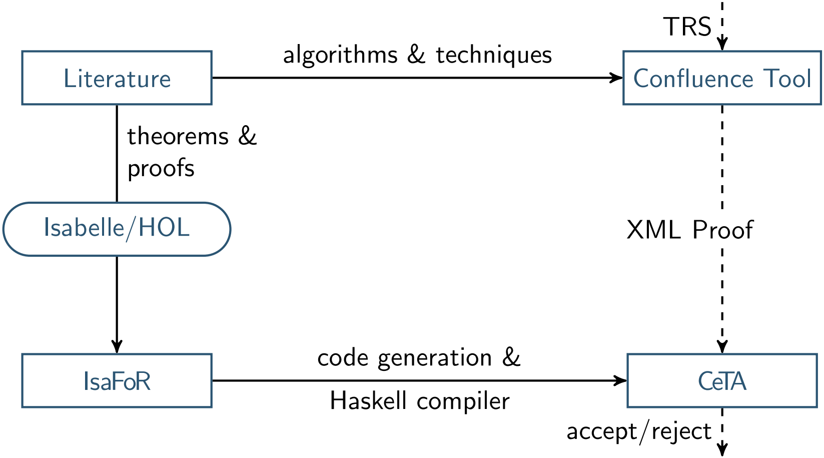

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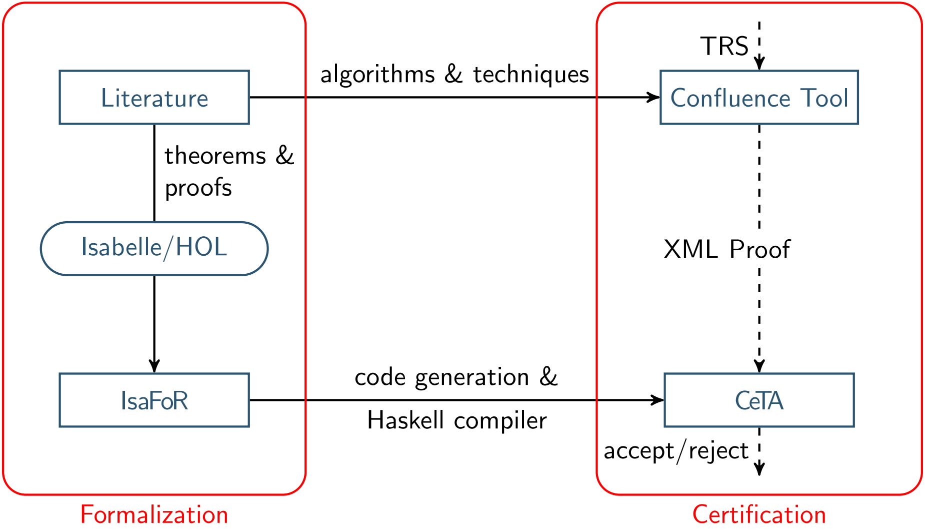

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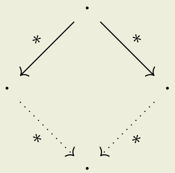

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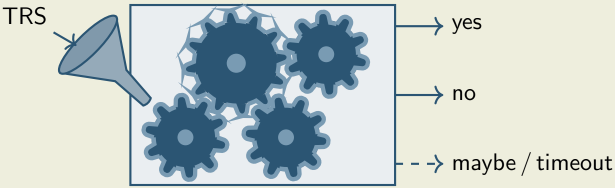

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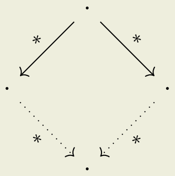

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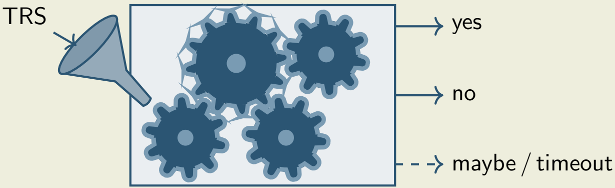

Strongly Closed Critical Pairs

Definition

$\rightarrow$ is strongly confluent if $\leftarrow \cdot \rightarrow \subseteq \rightarrow^* \cdot {}^{=}\!\!\leftarrow$

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TRS is strongly closed if $s \to^= \cdot {}^*\!\leftarrow t$ and $s \to^* \cdot {}^=\!\leftarrow t$ for every critical pair $t \leftarrow\!\rtimes\!\to s$

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Lemma

For linear term t, position $p \in \mathcal{P}\mathrm{os}(t)$ with $t|_p = x$ and substitutions σ and τ with $\sigma(y) = \tau(y)$ for all $y \in \mathcal{V}\mathrm{ars}(t)$ such that $y \neq x$ we have $t\tau = t\sigma[\tau(x)]_p$

Lemma

If $s \to_{\ell_1 \to r_1, p_1, \sigma_1} t$ and $s \to_{\ell_2 \to r_2, p_2, \sigma_2} u$ with $p_1 \leqslant p_2$ in a linear, strongly closed TRS there are terms v and w with $t \to^* v \;{}^=\!\!\leftarrow u$ and $t \to^= w \;{}^*\!\!\leftarrow u$

Proof

- from $p_1 \leqslant p_2$ obtain position q with $p_2 = p_1 q$ and $(\ell_1\sigma_1)|_q = \ell_2\sigma_2$
- $u = s[(\ell_1\sigma_1)[r_2\sigma_2]_q]_{p_1}$

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- case analysis on $q \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_1)$

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- case analysis on $q \in \mathcal{P}\text{os}_{\mathcal{F}}(\ell_1)$
- if $q \in \mathcal{P}\text{os}_{\mathcal{F}}(\ell_1)$ then $\ell_1|_q\sigma_1 = \ell_2\sigma_2$ and thus $\ell_1\mu[r_2\mu]_q \leftarrow\!\rtimes\!\rightarrow r_1\mu$
- then $r_1\mu \rightarrow^*_{\mathcal{R}} v \; {}_{\mathcal{R}}^{=}\!\!\leftarrow \ell_1\mu[r_2\mu]_q$ and $r_1\mu \rightarrow^{=}_{\mathcal{R}} w \; {}_{\mathcal{R}}^{*}\!\!\leftarrow \ell_1\mu[r_2\mu]_q$ by assumption
- closure under context and substitution yields result

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- closure under context and substitution yields result
- if $q \notin \mathcal{P}\mathrm{os}_{\mathcal{F}}(\ell_1)$ obtain positions q_1, q_2 and variable x with $q = q_1q_2$, $q_1 \in \mathcal{P}\mathrm{os}(\ell_1)$ $\ell_1|_{q_1} = x$, and $(x\sigma_1)|_{q_2} = \ell_2\sigma_2$
- define τ as
$$\tau(y) = \begin{cases} (x\sigma_1)[r_2\sigma_2]_{q_2} & \text{if } y = x \\ y\sigma_1 & \text{otherwise} \end{cases}$$
- since ℓ_1 is linear we have $\ell_1\tau = (\ell_1\sigma_1)[(x\sigma_1)[r_2\sigma_2]_{q_2}]_{q_1}$ using Lemma
- hence also $\ell_1\tau = (\ell_1\sigma_1)[r_2\sigma_2]_q$ and thus $u = s[\ell_1\tau]_{p_1} \to_{\mathcal{R}} s[r_1\tau]_{p_1}$

Proof (cont)

- show $t \to_{\mathcal{R}}^{=} s[r_1\tau]_{p_1}$

Proof (cont)

- show $t \to_{\mathcal{R}}^{=} s[r_1\tau]_{p_1}$, if $x \notin \mathcal{V}\mathsf{ars}(r_1)$ then $r_1\tau = r_1\sigma_1$ and thus $t = s[r_1\tau]_{p_1}$

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- if $x \in \mathcal{V}\mathsf{ars}(r_1)$ obtain position $q' \in \mathcal{P}\mathsf{os}(r_1)$ with $r_1|_{q'} = x$
- since r_1 is linear $r_1\tau = (r_1\sigma_1)[(x\sigma_1)[r_2\sigma_2]_{q_2}]_{q'}$ and hence $r_1\tau = (r_1\sigma_1)[r_2\sigma_2]_{q'q2}$
- since also $r_1\sigma_1 = (r_1\sigma_1)[\ell_2\sigma_2]_{q'q2}$ we have $r_1\sigma_1 \rightarrow_{\mathcal{R}} r_1\tau$ and thus also $t \rightarrow_{\mathcal{R}} s[r_1\tau]_{p_1}$

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Corollary (Huet)

If $\mathcal{R}$ is linear and strongly closed then $\rightarrow_{\mathcal{R}}$ is strongly confluent

Proof (cont)

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Corollary (Huet)

If $\mathcal{R}$ is linear and strongly closed then $\to_{\mathcal{R}}$ is strongly confluent

Proof

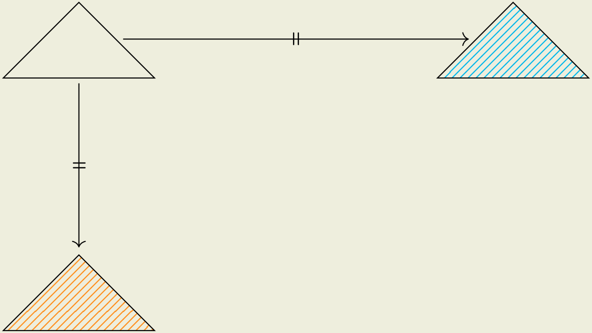
- assume $s \to_{\ell_1 \to r_1, p_1, \sigma_1} t$ and $s \to_{\ell_2 \to r_2, p_2, \sigma_2} u$
- show $t \to^* \cdot {}^{=}\!\!\leftarrow u$ by case analysis on p_1 and p_2
- if they are parallel then $t \to t[r_2\sigma_2]_{p_2} = u[r_1\sigma_1]_{p_1} \leftarrow u$
- if $p_1 \geqslant p_2$ or $p_2 \geqslant p_1$ by Lemma

Parallel Closed Critical Pairs

Theorem (Huet)

If $\mathcal{R}$ is left-linear and $t \mathrel{+\!\!\!\!\!\twoheadrightarrow} s$ for all $t \leftarrow\!\rtimes\!\rightarrow s$ then $\mathrel{+\!\!\!\!\!\twoheadrightarrow}$ has the diamond property

Proof by Picture



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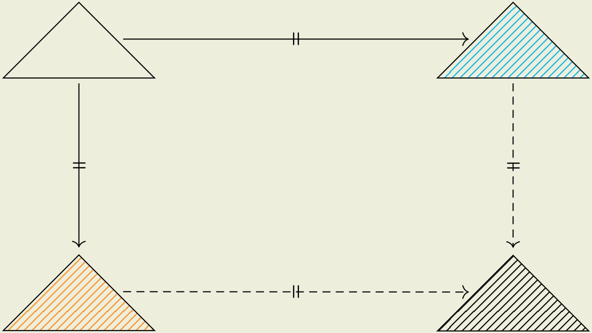

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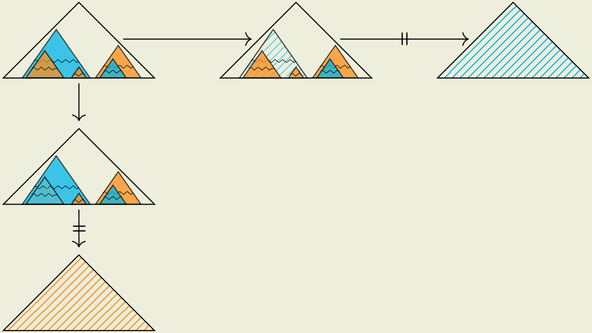

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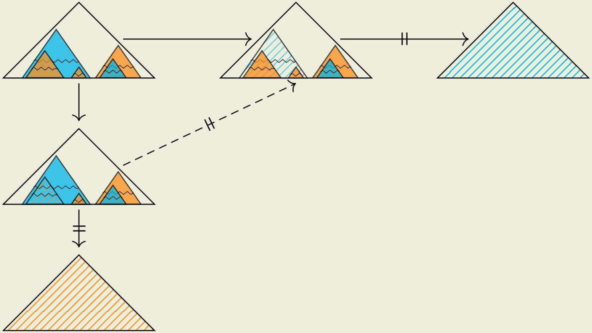

Parallel Closed Critical Pairs

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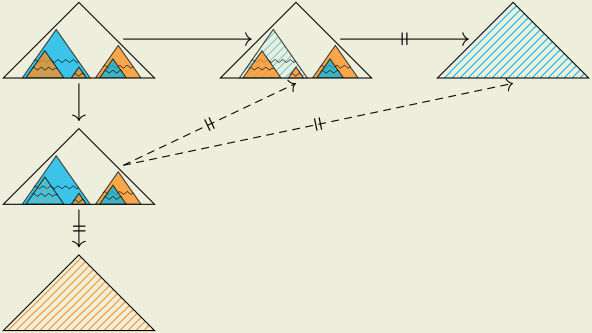
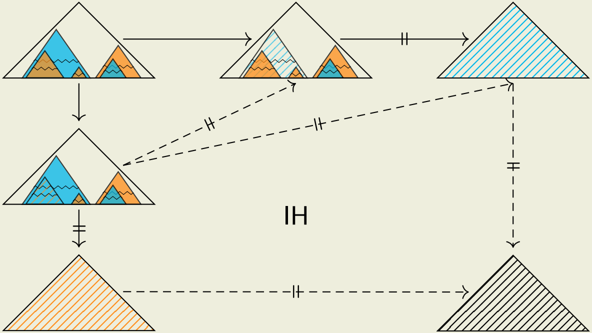


Parallel Closed Critical Pairs

Theorem (Huet)

If $\mathcal{R}$ is left-linear and $t \mathrel{-\!\!\!\parallel\!\!\!\twoheadrightarrow} s$ for all $t \leftarrow\!\rtimes\!\rightarrow s$ then $\mathrel{-\!\!\!\parallel\!\!\!\twoheadrightarrow}$ has the diamond property

Proof by Picture



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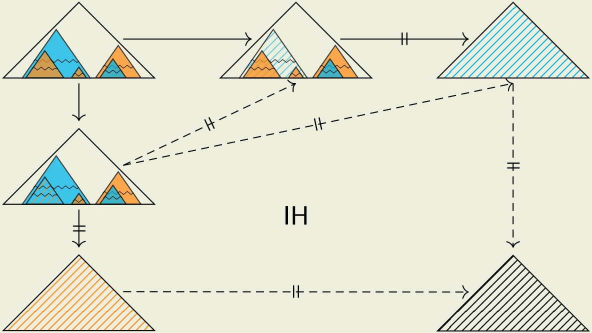

- how to represent parallel rewriting?

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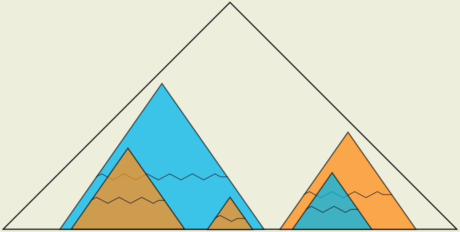

- how to measure "amount of overlap"?

Parallel Closed Critical Pairs

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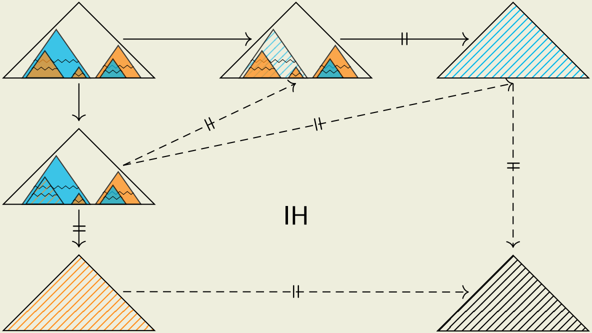

- positions and multihole contexts

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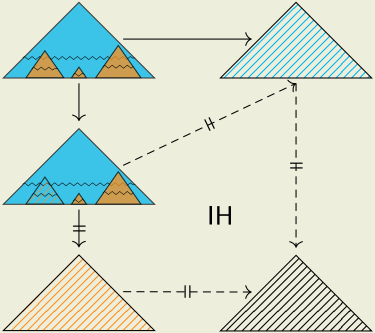

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Parallel Rewriting and Overlap

Definition

$s \xrightarrow[\parallel]{C, s_1, \ldots, s_n} t$ if $s = C[s_1, \ldots, s_n]$, $t = C[t_1, \ldots, t_n]$ and $s_i \rightarrow_\epsilon t_i$ for all $1 \leqslant i \leqslant n$

Parallel Rewriting and Overlap

Definition

$s \xrightarrow{C,\overline{s}}\!\!\!\!\!\!\!+\!\!\!+\; t$ if $s = C[s_1, \ldots, s_n]$, $t = C[t_1, \ldots, t_n]$ and $s_i \rightarrow_\epsilon t_i$ for all $1 \leqslant i \leqslant n$

Definition

Overlap between parallel steps $\xrightarrow{C,\overline{s}}\!\!\!\!\!\!\!+\!\!\!+\;$ and $\xrightarrow{D,\overline{t}}\!\!\!\!\!\!\!+\!\!\!+\;$ is
$\blacktriangle(C, \overline{s}, D, \overline{t}) = \{p \mid p \notin \mathcal{P}\mathrm{os}(C) \wedge p \notin \mathcal{P}\mathrm{os}(D) \wedge p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(C[\overline{s}]) \wedge \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(D[\overline{t}])\}$

Parallel Rewriting and Overlap

Definition

$s \xrightarrow{C,\overline{s}}\!\!\!\!\!\!\!\!\!\!\!\!\;\;\;\;\;\;\;\;\;\; t$ if $s = C[s_1, \ldots, s_n]$, $t = C[t_1, \ldots, t_n]$ and $s_i \to_\epsilon t_i$ for all $1 \leqslant i \leqslant n$

Definition

Overapproximation of overlap between parallel steps $\overset{C,\overline{s}}{\dashv\!\vdash\!\!\rightarrow}$ and $\overset{D,\overline{t}}{\dashv\!\vdash\!\!\rightarrow}$ is
$\blacktriangle(C, \overline{s}, D, \overline{t}) = \{p \mid p \notin \mathcal{P}\mathrm{os}(C) \wedge p \notin \mathcal{P}\mathrm{os}(D) \wedge p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(C[\overline{s}]) \wedge \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(D[\overline{t}])\}$

Parallel Rewriting and Overlap

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$s \xrightarrow{C,\bar{s}}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\;\;\;\;\;\;\;\;\;\;\;\;\;\;\; t$ if $s = C[s_1, \ldots, s_n]$, $t = C[t_1, \ldots, t_n]$ and $s_i \to_\epsilon t_i$ for all $1 \leqslant i \leqslant n$

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Example

$\mathcal{R} : \mathsf{f}(\mathsf{a}, \mathsf{b}) \to \mathsf{f}(\mathsf{a}, \mathsf{a})$

$\mathsf{a} \to \mathsf{b} \quad \mathsf{b} \to \mathsf{a}$

$\mathsf{f}(\mathsf{a}, \mathsf{b})$

Parallel Rewriting and Overlap

Definition

$s \overset{C,\overline{s}}{-\!\!\Vert\!\!\!\rightarrow} t$ if $s = C[s_1, \ldots, s_n]$, $t = C[t_1, \ldots, t_n]$ and $s_i \to_\epsilon t_i$ for all $1 \leqslant i \leqslant n$

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Overapproximation of overlap between parallel steps $\overset{C,\overline{s}}{-\!\!\Vert\!\!\!\rightarrow}$ and $\overset{D,\overline{t}}{-\!\!\Vert\!\!\!\rightarrow}$ is
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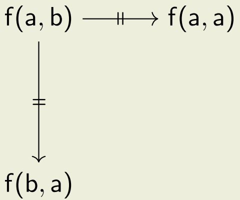
Example

$$\mathcal{R} : \mathsf{f}(\mathsf{a},\mathsf{b}) \to \mathsf{f}(\mathsf{a},\mathsf{a}) \qquad \mathsf{a} \to \mathsf{b} \quad \mathsf{b} \to \mathsf{a}$$

$$\blacktriangle(\Box, [\mathsf{f}(\mathsf{a},\mathsf{b})], \mathsf{f}(\Box,\Box), [\mathsf{a},\mathsf{b}]) = \{1, 2\}$$



Parallel Rewriting and Overlap

Definition

$s \xrightarrow[\Vert]{C,\overline{s}} t$ if $s = C[s_1, \ldots, s_n]$, $t = C[t_1, \ldots, t_n]$ and $s_i \rightarrow_\epsilon t_i$ for all $1 \leqslant i \leqslant n$

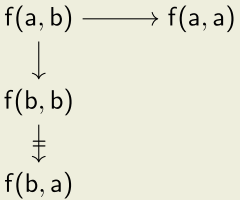
Definition

Overapproximation of overlap between parallel steps $\xrightarrow[\Vert]{C,\overline{s}}$ and $\xrightarrow[\Vert]{D,\overline{t}}$ is
$\blacktriangle(C, \overline{s}, D, \overline{t}) = \{p \mid p \notin \mathcal{P}\mathrm{os}(C) \wedge p \notin \mathcal{P}\mathrm{os}(D) \wedge p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(C[\overline{s}]) \wedge \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(D[\overline{t}])\}$

Example

$$\mathcal{R} : \mathsf{f}(\mathsf{a}, \mathsf{b}) \rightarrow \mathsf{f}(\mathsf{a}, \mathsf{a})$$
$$\mathsf{a} \rightarrow \mathsf{b} \quad \mathsf{b} \rightarrow \mathsf{a}$$

$$\blacktriangle(\square, [\mathsf{f}(\mathsf{a}, \mathsf{b})], \mathsf{f}(\square, \square), [\mathsf{a}, \mathsf{b}]) = \{1, 2\}$$



Parallel Rewriting and Overlap

Definition

$s \overset{C,\overline{s}}{-\!\!\!\Vert\!\!\!\rightarrow} t$ if $s = C[s_1, \ldots, s_n]$, $t = C[t_1, \ldots, t_n]$ and $s_i \rightarrow_\epsilon t_i$ for all $1 \leqslant i \leqslant n$

Definition

Overapproximation of overlap between parallel steps $\overset{C,\overline{s}}{-\!\!\!\Vert\!\!\!\rightarrow}$ and $\overset{D,\overline{t}}{-\!\!\!\Vert\!\!\!\rightarrow}$ is
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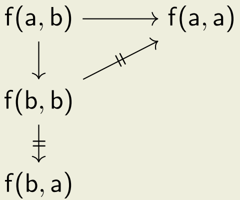
$$\blacktriangle(\mathsf{f}(\square, \square), [\mathsf{b}, \mathsf{b}], \mathsf{f}(\mathsf{b}, \square), [\mathsf{b}]) = \{2\}$$



Lemma

For linear s with $s\sigma = C[s_1, \ldots, s_n] \mathrel{\|\!\!\!\rightarrow} C[t_1, \ldots, t_n] = t$ there is τ with either

- *$t = s\tau$ and $x\sigma \mathrel{\|\!\!\!\rightarrow} x\tau$ for all $x \in \mathcal{V}\mathrm{ars}(s)$, or*
- *$s = D[s']$ for a context D and non-variable term s' and there is a rule $\ell \to r$ such that $s'\sigma = \ell\tau = s_i$, $r\tau = t_i$*

for some $1 \leqslant i \leqslant n$

$$s\sigma = C[s_1, \ldots, s_n] \xrightarrow{\qquad\qquad \| \qquad\qquad} t$$

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for some $1 \leqslant i \leqslant n$

$$s\sigma = C[s_1, \ldots, s_n] \mathrel{+\!\!\!\!\!\longrightarrow} t$$

Lemma

For linear s with $s\sigma = C[s_1, \ldots, s_n] \nrightarrow\!\!\!\!\rightarrow C[t_1, \ldots, t_n] = t$ there is τ with either

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  for some $1 \leqslant i \leqslant n$

$$s\sigma = C[s_1, \ldots, s_n] \longrightarrow D\sigma[r\tau] \qquad\qquad t$$

Lemma

For linear s with $s\sigma = C[s_1, \ldots, s_n] \mathrel{+\!\!\!\!\!\twoheadrightarrow} C[t_1, \ldots, t_n] = t$ there is τ with either

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$$s\sigma = C[s_1, \ldots, s_n] \longrightarrow D\sigma[r\tau] \mathrel{\longrightarrow\!\!\!\!\!\!\!\!\parallel\;\;} t$$

Lemma

For linear s with $s\sigma = C[s_1, \ldots, s_n] \mathrel{\Vert\!\!\!\to} C[t_1, \ldots, t_n] = t$ there is τ with either

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$$\begin{array}{ccccc} s\sigma = C[s_1, \ldots, s_n] & \longrightarrow & D\sigma[r\tau] & \mathrel{-\!\!\Vert\!\!\!\longrightarrow} & t \\ \downarrow & & & & \\ u & & & & \end{array}$$

Theorem (Huet)

If $\mathcal{R}$ is left-linear and $t \twoheadrightarrow\!\!\!\!\!\!\shortmid\,\, s$ for all $t \leftarrow\!\rtimes\!\rightarrow s$ then $\twoheadrightarrow\!\!\!\!\!\!\shortmid\,\,$ has the diamond property

Proof

- assume $s \xrightarrow[\shortparallel]{C,\overline{s^c}} t$ and $s \xrightarrow[\shortparallel]{D,\overline{s^d}} u$, nested induction on $|\blacktriangle(C, \overline{s^c}, D, \overline{s^d})|$ and s

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- if $s = x$ then $t = u = x$

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- if $s = x$ then $t = u = x$
- let $s = f(s_1, \ldots, s_n)$, case analysis on C and D
- case $C = f(c_1, \ldots, c_n)$ and $D = f(d_1, \ldots, d_n)$, then $t = f(t_1, \ldots, t_n)$ and $u = f(u_1, \ldots, u_n)$

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- then $\overline{s^c}$ and $\overline{s^d}$ can be partitioned into $ss^c_1, \ldots, ss^c_n$ and $ss^d_1, \ldots, ss^d_n$ such that $s_i \xrightarrow[]{c_i, ss^c_i}\!\!\!\!\!\!\shortparallel\ \ t_i$ and $s_i \xrightarrow[]{d_i, ss^d_i}\!\!\!\!\!\!\shortparallel\ \ u_i$ for all $1 \leqslant i \leqslant n$

Theorem (Huet)

If $\mathcal{R}$ is left-linear and $t \mathrel{+\!\!\!\!\twoheadrightarrow} s$ for all $t \leftarrow\!\rtimes\!\rightarrow s$ then $\mathrel{+\!\!\!\!\twoheadrightarrow}$ has the diamond property

Proof

- assume $s \xrightarrow{C,\overline{s^c}}_{\Vert} t$ and $s \xrightarrow{D,\overline{s^d}}_{\Vert} u$, nested induction on $|\blacktriangle(C, \overline{s^c}, D, \overline{s^d})|$ and s
- if $s = x$ then $t = u = x$
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- moreover $|\blacktriangle(c_i, ss_i^c, d_i, ss_i^d)| \leqslant |\blacktriangle(C, \overline{s^c}, D, \overline{s^d})|$ for all $1 \leqslant i \leqslant n$

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- moreover $|\blacktriangle(c_i,ss^c_i,d_i,ss^d_i)| \leqslant |\blacktriangle(C,\overline{s^c},D,\overline{s^d})|$ for all $1 \leqslant i \leqslant n$
- hence there are v_i with $t_i \twoheadrightarrow\!\!\!\!\!\!\parallel\; v_i \;\parallel\!\!\!\!\!\!\twoheadleftarrow u_i$ for all $1 \leqslant i \leqslant n$ by inner IH
- thus $t \twoheadrightarrow\!\!\!\!\!\!\parallel\; v \;\parallel\!\!\!\!\!\!\twoheadleftarrow u$ for $v = f(v_1,\ldots,v_n)$

Proof

- assume $C = f(c_1, \ldots, c_n)$ and $D = \Box$
- so $s = \ell\sigma$ and $u = r\sigma$ for some $\ell \to r \in \mathcal{R}$

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- then by Lemma either $t = \ell\tau$ and $x\sigma \mathrel{+\!\!\!\!\!\!\twoheadrightarrow} x\tau$ for all $x \in \mathcal{V}\mathrm{ars}(s)$, or there is a critical pair

Proof

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- then by Lemma either $t = \ell\tau$ and $x\sigma \nrightarrow\!\!\!\!\!\rightarrow x\tau$ for all $x \in \mathcal{V}\mathrm{ars}(s)$, or there is a critical pair
- in the first case let

$$\delta(x) = \begin{cases} \tau(x) & \text{if } x \in \mathcal{V}\mathrm{ars}(\ell) \\ \sigma(x) & \text{otherwise} \end{cases}$$

- then $t = \ell\tau = \ell\delta \nrightarrow\!\!\!\!\!\rightarrow r\delta \nleftarrow\!\!\!\!\!\leftarrow r\sigma = u$

Proof

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- then by Lemma either $t = \ell\tau$ and $x\sigma \nrightarrow\!\!\!\!\!\nrightarrow x\tau$ for all $x \in \mathcal{V}\mathrm{ars}(s)$, or there is a critical pair
- if there is a critical pair write $\ell = E[\ell'']$ and obtain a rule $\ell' \to r'$ such that $\ell''\sigma = \ell'\tau = s_i^c$, $r'\tau = t_i^c$ and $E\sigma = C[s_1^c, \ldots, s_{i-1}^c, \Box, s_{i+1}^c, \ldots, s_n^c]$, $E\sigma[r'\tau] = C[\Box, \ldots, \Box, t_i^c, \Box, \ldots, \Box][s_1^c, \ldots, s_{i-1}^c, s_{i+1}^c, \ldots, s_n^c]$, and $t = C[\Box, \ldots, \Box, t_i^c, \Box, \ldots, \Box][t_1^c, \ldots, t_{i-1}^c, t_{i+1}^c, \ldots, t_n^c]$ for some $1 \leqslant i \leqslant n$

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- $E\mu[r'\mu] \leftarrow\!\rtimes\!\rightarrow r\mu$ is closed $E\mu[r'\mu] \twoheadrightarrow\!\!\!\!\!\!+ r\mu$ by assumption
- then also $E\sigma[r'\tau] \xrightarrow{F,\overline{f}}\!\!\!\!\!\!\!\!\!+\;\; r\sigma$ for some F, $\overline{f}$

Proof

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- then also $E\sigma[r'\tau] \xrightarrow{F,\overline{f}}\!\!\!\!\!\!\!\!+\ \ r\sigma$ for some F, $\overline{f}$
- to apply outer induction hypothesis show $\blacktriangle(C[\Box, \ldots, \Box, t_i^c, \Box, \ldots, \Box], [s_1^c, \ldots, s_{i-1}^c, s_{i+1}^c, \ldots, s_n^c], F, \overline{f}) \subset \blacktriangle(C, \overline{s^c}, \Box, [\ell\sigma])$

Proof

- since $E\sigma[r'\tau] = F[\overline{f}]$ a position in $F[\overline{f}]$ is either in $E\sigma$ and thus in $\ell\sigma$ or below the hole of E and thus in $C[\square, \ldots, \square, t_i^c, \square, \ldots, \square]$

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- moreover positions in $C[\square, \ldots, \square, t_i^c, \square, \ldots, \square][s_1^c, \ldots, s_{i-1}^c, s_{i+1}^c, \ldots, s_n^c]$ that are not in $C[\square, \ldots, \square, t_i^c, \square, \ldots, \square]$ are also in $C[\overline{s^c}]$ but not in C
- $\blacktriangle(C[\square, \ldots, \square, t_i^c, \square, \ldots, \square], [s_1^c, \ldots, s_{i-1}^c, s_{i+1}^c, \ldots, s_n^c], F, \overline{f}) \subseteq \blacktriangle(C, \overline{s^c}, \square, [\ell\sigma])$

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- $\blacktriangle(C[\Box, \ldots, \Box, t_i^c, \Box, \ldots, \Box], [s_1^c, \ldots, s_{i-1}^c, s_{i+1}^c, \ldots, s_n^c], F, \overline{f}) \subseteq \blacktriangle(C, \overline{s^c}, \Box, [\ell\sigma])$
- additionally the hole position of E is in $\blacktriangle(C, \overline{s^c}, \Box, [\ell\sigma])$ since it is a function position of $\ell\sigma$ and not in C but in $C[\overline{s^c}]$
- but since it is in $C[\Box, \ldots, \Box, t_i^c, \Box, \ldots, \Box]$, it is not in $\blacktriangle(C[\Box, \ldots, \Box, t_i^c, \Box, \ldots, \Box], [s_1^c, \ldots, s_{i-1}^c, s_{i+1}^c, \ldots, s_n^c], F, \overline{f})$

Proof

- since $E\sigma[r'\tau] = F[\overline{f}]$ a position in $F[\overline{f}]$ is either in $E\sigma$ and thus in $\ell\sigma$ or below the hole of E and thus in $C[\square, \dots, \square, t_i^c, \square, \dots, \square]$
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- $\blacktriangle(C[\square, \dots, \square, t_i^c, \square, \dots, \square], [s_1^c, \dots, s_{i-1}^c, s_{i+1}^c, \dots, s_n^c], F, \overline{f}) \subseteq \blacktriangle(C, \overline{s^c}, \square, [\ell\sigma])$
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- hence there is a v such that $t \twoheadrightarrow\!\!\!\!\!\!\!+\; v \;\;+\!\!\!\!\!\!\!\twoheadleftarrow r\sigma$ by outer IH

Proof

- since $E\sigma[r'\tau] = F[\overline{f}]$ a position in $F[\overline{f}]$ is either in $E\sigma$ and thus in $\ell\sigma$ or below the hole of E and thus in $C[\square, \ldots, \square, t_i^c, \square, \ldots, \square]$
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- additionally the hole position of E is in $\blacktriangle(C, \overline{s^c}, \square, [\ell\sigma])$ since it is a function position of $\ell\sigma$ and not in C but in $C[\overline{s^c}]$
- but since it is in $C[\square, \ldots, \square, t_i^c, \square, \ldots, \square]$, it is not in $\blacktriangle(C[\square, \ldots, \square, t_i^c, \square, \ldots, \square], [s_1^c, \ldots, s_{i-1}^c, s_{i+1}^c, \ldots, s_n^c], F, \overline{f})$
- hence there is a v such that $t \twoheadrightarrow\!\!\!\!\!\!\| \; v \twoheadleftarrow\!\!\!\!\!\!\| \; r\sigma$ by outer IH
- case $D = f(d_1, \ldots, d_n)$ and $C = \square$ is completely symmetric
- case $D = C = \square$ is simpler: since both steps are single root steps, closing the resulting CP closes the whole peak

Almost Parallel Closed Critical Pairs

Theorem (Toyama)

If $\mathcal{R}$ is left-linear, $t \mathrel{+\!\!\!\!\!\!\;+\!\!\!\!\!\rightarrow} s$ for all inner critical pairs $t \leftarrow\!\!\cdot\!\!\rtimes\!\!\rightarrow s$, and $t \mathrel{+\!\!\!\!\!\!\;+\!\!\!\!\!\rightarrow} \cdot {}^{*}\!\!\leftarrow s$ for all overlays $t \leftarrow\!\bowtie\!\rightarrow s$ then $\mathrel{+\!\!\!\!\!\!\;+\!\!\!\!\!\rightarrow}$ is strongly confluent

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Proof (Adaptations)

- $s \xrightarrow{C,\overline{s^c}}\!\!\!\!\!\!\!\!\!\!\!\!\Vert\ \ \ \ \ t$ and $s \xrightarrow{D,\overline{s^d}}\!\!\!\!\!\!\!\!\!\!\!\!\Vert\ \ \ \ \ u$
- prove $t \mathrel{+\!\!\!\!\!\!\twoheadrightarrow}^* \cdot \twoheadleftarrow\!\!\!\!\!\!+\ u$ and $u \mathrel{+\!\!\!\!\!\!\twoheadrightarrow}^* \cdot \twoheadleftarrow\!\!\!\!\!\!+\ t$
- if $C = D = \Box$ then assumption for overlays applies
- other cases remain (almost) the same

Development Closed Critical Pairs

Theorem (van Oostrom)

If $\mathcal{R}$ is left-linear and $t \mathrel{+\!\!\!\!\!\twoheadrightarrow} s$ for all critical peaks $t \leftarrow\!\rtimes\!\rightarrow s$ then $\mathrel{+\!\!\!\!\!\twoheadrightarrow}$ has the diamond property

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- nesting of steps makes describing $\mathrel{\circ\!\!\!\!\rightarrow\!\!\!\!\rightarrow}$-steps harder
- induction on source of peak does not help
- need to split off single steps on both sides and combine closing step with remainder
- due to nesting of redexes this needs non-trivial reasoning about residuals
- need to split off "innermost" overlap to get decrease in measure
- overapproximation of overlap does not work

Summary

- first formalization of two classical confluence results
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- combination of multihole contexts and positions
- multihole contexts for describing steps
- positions in decomposed steps for measuring amount of overlap

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Differences to Paper Proof

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- combination of multihole contexts and positions
- multihole contexts for describing steps
- positions in decomposed steps for measuring amount of overlap

- future work: development closed
- harder future work: apply to higher-order rewriting